

Linear Algebra Primer: Part 4- Functions

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Functions

In this section, Functions will be introduced. Much of this terminology is fairly basic, but critical to upper level mathematics, including linear algebra. Thus, this section will precede the section on linear transformations to provide sufficient background on functions.

Functions are introduced in lower-level mathematics, often lacking the formality. This section will add the formality and some terminology.

A Function is a mapping between two sets, known as the Domain and Codomain. The Domain is the input set, and the Codomain is the set to which the Domain is mapped. The Range is the subset of the codomain whose elements are actually mapped. The codomain can contain unmapped elements. All functions map uniquely. That means that no element in the domain can map to more than one element in the codomain. However, a single element in the codomain can have multiple values in the domain map to it.

Consider the following function: $f : \mathbb{R} \rightarrow \mathbb{R}$, such that $f(x) = x^2$. It's easy to see why f is a function. Notice that the codomain is the set of real numbers. However, since the domain is limited to the real numbers, the range is set of the non-negative reals. Thus, the range is a subset of the codomain.

Functions can also be one-to-one, onto, or both. These terms are important in developing an intuition about linear transformations, and provide a lot of tools including the isomorphisms and equivalence relations.

A function is **one-to-one** if $\forall x, y \in D$, where D is the domain, $f(x) = f(y)$ only when $x = y$. An example of a one-to-one function is $f(x) = 5x$. Consider as well the sets $X = \{A, B, C\}$ and $Y = \{1, 2, 3, 4\}$ with $F : X \rightarrow Y$ defined by:

$$\begin{aligned}F(A) &= 1 \\F(B) &= 2 \\F(C) &= 3.\end{aligned}$$

Note that this is a one-to-one function. Every element in the domain maps uniquely to an element in the codomain. Notice though that there are unmapped elements in the codomain. In a domain D , a function can have at most $|D|$ distinct mappings, which is achieved in a one-to-one function. It makes no guarantees that all the elements in the codomain are mapped. Thus, a one-to-one function guarantees that the cardinality of the codomain is no smaller than the cardinality of the domain.

A function is considered **onto** if all the elements in the codomain are mapped. More precisely, $\forall y \in C$, where C is the codomain, $\exists x \in D$, the domain, such that $f(x) = y$. Consider the example $f : \mathbb{R} \rightarrow \mathbb{R}$ by the mapping $f(x) = x^3$. It is easy to see by graphing the function

that it is onto, as it covers the entire real number system. However, if f was defined such that $f : \mathbb{Q} \rightarrow \mathbb{R}$, then the function would not be onto, as plugging in only rationals will not cover the entire set of reals.

Consider another example with the sets $X = \{A, B, C\}$ and $Y = \{1, 2, 3, 4\}$ with $F : Y \rightarrow X$:

$$F(1) = A$$

$$F(2) = B$$

$$F(3) = C$$

$$F(4) = A$$

This function is onto, as it covers the entire codomain, which is set X . However, it is not one-to-one. Thus, if a function is onto, then $|Domain| \geq |Codomain|$.

Finally, a function is a **bijection** if it is both one-to-one and onto. As a one-to-one function guarantees that $|Domain| \leq |Codomain|$ and an onto function guarantees that $|Domain| \geq |Codomain|$, a bijection guarantees that $|Domain| = |Codomain|$. For this reason, bijections are commonly used in combinatorial applications for counting. This combinatorial intuition, once developed, is highly applicable to answering questions regarding linear transformations. For now though, let's introduce a couple examples of bijections. The function seen earlier $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^3$ is a bijection. It covers all the real numbers, so it is onto, and it maps elements in the domain uniquely to elements in the codomain.

Consider as well: $f : \mathbb{Z} \rightarrow \mathbb{Z}_{even}$, so mapping the integers to the even integers by $f(x) = 2x$. Even integers are defined such that $n = 2x$, where $x \in \mathbb{Z}$. And $2x = 2y$ only when $x = y$, so this function is one-to-one and onto. Looking at the function the other way, $f^{-1} : \mathbb{Z}_{even} \rightarrow \mathbb{Z}$ by $f^{-1}(y) = \frac{y}{2}$, it still produces a unique integer for every even integer. Thus it can be concluded that there are as many even integers as integers, as they have the same count.

When dealing with Linear Transformations, infinite sets come into play, as Vector Spaces are often times infinite in cardinality, just like the integers. So it becomes important to see bijections in this manner described above. This will be discussed more in the next section, however.

Finally, let's introduce the concept of **invertibility**. An invertible function is defined such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. This is called the identity property. By definition of an inverse, if the function $f : X \rightarrow Y$, then $f^{-1} : Y \rightarrow X$. A function is invertible only if it is a bijection. Let's examine why this is. Consider if a function is strictly one-to-one; that is, one-to-one, but not onto. This leaves the possibility for unmapped elements in the codomain. The inverse function must map all the elements of the original codomain to elements of the original domain, by definition of a function.

Similarly, if a function is strictly onto, then two elements in the domain can map to the same element in the codomain. So the possibility exists for $f(x) = f(y)$ even if $x \neq y$. Thus, the inverse function could map an element in the original function's codomain to more than one element in the original function's domain, which violates the definition of a function.

From this analysis, in order for a function to be invertible, each element in the domain must map uniquely to an element in the codomain; and each element in the codomain must be mapped. Thus, the function must be bijective.

So $f(x) = x^3$ is invertible as it is bijective and thus invertible by $f^{-1}(x) = x^{\frac{1}{3}}$, but $f(x) = x^2$ is not invertible, as the square root function is only defined on the domain of non-negative real

numbers, thus it is not onto.